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DETERMINATION BY TEST PUMPING OF VARIABLE PERMEABILITY OF A STRATUM UNDER CONDITIONS OF RADIAL SYMMETRY

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A central borehole in a circular stratum whose permeability depends only on the radius is considered. The permeability coefficient is determined by the flow rate and the pressure in the borehole with constant pressure at the contour of the latter. The problem of determination of the permeability coefficient reduces to the restitution of the Sturm-Liouville operator over its spectral function. Nominal correctness of the problem is proved in the case in which the permeability coefficient belongs to the class of bounded positive functions with bounded first and second derivatives.

Similar problems for hyperbolic equations were considered in [1, 2]. The theorem of the uniqueness of certain inverse problems was proved in [3] for parabolic equations in a similar formulation. The ideas and methods applied here are closest to those in [1].

1. On the uniqueness of restitution of the permeability coefficient in the equation of filtration by an overdetermined system of boundary conditions. Let us consider the inverse problem of the particular case of parabolic equations which occur in the theory of filtration in determining the variable permeability coefficient by actual test data.

Let us consider a circular stratum of radius R with a central borehole of radius r_0 and assume that the stratum permeability is variable and dependent only on radius r. We have to determine the permeability on the basis of observations of the borehole operation conditions. We assume that at the initial instant of time $t = 0$ the borehole is not working and that the pressure throughout the stratum is $p_{t=0} = 0$. Pumping of petroleum through the borehole then begins at the rate $q(t)$.

Since function $q(t)$ can be directly measured, we assume it to be known. The related pressure in the borehole can also be measured. Pressure variation in the stratum is defined by the equation

$$\beta \frac{\partial p}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[rk(r) \frac{\partial p}{\partial r} \right] \tag{1.1}$$

where β is a certain constant, and by the boundary conditions

$$p_{t=0} = 0, \quad 2\pi r_0 k(r_0) \frac{\partial p}{\partial r}\Big|_{r=r_0} = q(t), \quad p_{r=R} = 0 \tag{1.2}$$

Let the supplementary conditions

$$p_{r=r_0} = \varphi(t) \tag{1.3}$$

be also specified.

Let us prove that, if there exists a positive function $k(r)$ which is continuously differentiable along segment $[r_0, R]$ and which satisfies (1.1)–(1.3), then that function is unique.

We seek the solution of Eq. (1.1) with boundary conditions (1.2) in the form of series in eigenfunctions of operator

$$LP = -\frac{\partial}{\partial r} \left[rk(r) \frac{\partial P}{\partial r} \right] = \lambda r P \tag{1.4}$$

$$P_{r=r_0} = 0, \qquad P_{r=R} = 0$$

We carry out the Laplace transformation of (1.1) and (1.2) with respect to time and introduce function

$$P^*(r, s) = P(r, s) - Q(s)\, \rho(r)$$

$$P(r, s) = \int_0^\infty p(r, t)\, e^{-st} dt, \qquad Q(s) = \int_0^\infty \frac{q(t)}{2\pi} e^{-st} dt$$

$$\rho(r) = \int_{r_0}^{r} \frac{dr}{rk(r)} - \int_{r_0}^{R} \frac{dr}{rk(r)}$$

Function $P^*(r, s)$ satisfies the equation

$$LP^* = -\beta rs\,(P^* + Q(s)\,\rho(r))$$

$$\frac{dP^*}{dr}\bigg|_{r=r_0} = 0, \qquad P^*(R) = 0$$

Let λ_k and $P_k(r)$ be eigenvalues and normalized eigenfunctions of operator (1.4). Then for $P^*(r, s)$ we obtain the expression

$$P^*(r,s) = -\sum_{k=1}^{\infty} \frac{\beta s Q(s)\,\mu_k}{\lambda_k + \beta s} P_k(r)$$

$$\mu_k = \int_{r_0}^{R} r\rho(r)\,P_k(r)\,dr = -\frac{1}{\lambda_k}\int_{r_0}^{R} \frac{\partial}{\partial r}\left[rk(r)\frac{\partial P_k}{\partial r}\right]\rho(r)\,dr$$

Constants μ_k are coefficients of the Fourier function $\rho(r)$ of system $P_k(r)$. Integrating twice by parts the right-hand side of the last equality and allowing for $L\rho = 0$ and $\rho(R) = 0$, we obtain $\mu_k = -P_k(r_0)/\lambda_k$

Then

$$P(r,s) = P^*(r,s) + Q(s)\rho(r) = -\sum_{k=1}^{\infty} \frac{Q(s)\,P_k(r_0)}{\lambda_k + \beta s} P_k(r) \tag{1.5}$$

Since $k(r) > 0$, series (1.5) is, by virtue of Mercer's theorem, regularly convergent in $[r_0, R]$. Setting in (1.5) $r = r_0$, we obtain

$$P(r_0, s) = \Phi(s) = -Q(s)\sum_{k=1}^{\infty} \frac{P_k^2(r_0)}{\lambda_k + \beta s} \tag{1.6}$$

$$\Phi(s) = \int_0^{\infty} \varphi(t)\,e^{-st}dt$$

We introduce the notation

$$\lambda = -\beta s, \quad \Phi^*(\lambda) = \Phi(-\lambda/\beta) = \Phi(s), \quad Q^*(\lambda) = Q(s)$$

With this from (1.6) we obtain

$$-\frac{\Phi^*(\lambda)}{Q^*(\lambda)} = \sum_{k=1}^{\infty} \frac{P_k^2(r_0)}{\lambda_k - \lambda} = \sum_{k=1}^{\infty} \frac{1}{\alpha_k(\lambda_k - \lambda)}, \quad \alpha_k = P_k^{-2}(r_0) \tag{1.7}$$

where α_k are normalization factors of problem (1.4). Thus the eigenvalues of operator (1.4) correspond to poles of function $\Phi^*(\lambda)/Q^*(\lambda)$, and the normalization factors are

$$1/\alpha_k = \mathrm{Res}_{\lambda=\lambda_k}(\Phi^*(\lambda)/Q^*(\lambda))$$

We carry out in Eq. (1.4) the following substitution of variables [5]

$$x = \frac{\pi}{B}\int_{r_0}^{r} \frac{dr}{\sqrt{k(r)}}, \qquad B = \int_{r_0}^{R} \frac{dr}{\sqrt{k(r)}} \tag{1.8}$$

$$r\sqrt{k(r)} = \theta(x), \qquad P\sqrt{\theta} = z(x)$$

Then $z(x)$ satisfies equation

$$-z'' + l(x)\,z = \gamma z \tag{1.9}$$

$$z'(0) - hz(0) = 0, \qquad z(\pi) = 0$$

$$l(x) = \frac{(\sqrt{\theta})''}{\sqrt{\theta}}, \quad \gamma = \frac{B^2}{\pi^2}\lambda, \quad h = \frac{\theta'(0)}{2\theta(0)} \tag{1.10}$$

We denote by γ_k and β_k, respectively, the eigenvalues and the normalization factors of problem (1.9)

$$\gamma_k = B^2\pi^{-2}\lambda_k, \qquad \beta_k = \pi\alpha_k / B\theta(0)$$

For γ_k and β_k are valid the asymptotic formulas [6]

$$\sqrt{\gamma_k} = k + \frac{1}{2} + O\left(\frac{1}{k}\right), \qquad \beta_k = \frac{\pi}{2} + O\left(\frac{1}{k}\right)$$

From this it is possible to determine constants

$$B = \pi \lim_{k\to\infty} \frac{k}{\sqrt{\lambda_k}}, \qquad \theta(0) = \frac{2}{B}\lim_{k\to\infty}\alpha_k \tag{1.11}$$

Thus the set $\{\gamma_k, \beta_k\}$ for the operator (1.9) is known. Hence we can determine function $l(x)$ by solving the inverse Sturm-Liouville problem for (1.9) [6, 7].

Let $\psi(x, \gamma)$ be a solution of Eq. (1.9) with initial conditions

$$\psi(0, \gamma) = 1, \quad \psi'(0, \gamma) = h \tag{1.12}$$

There exists a kernel $K(x, t)$ such that [6, 7]

$$\psi(x,\gamma) = \cos\sqrt{\gamma}\,x + \int_0^x K(x,t)\cos\sqrt{\gamma}t\,dt \tag{1.13}$$

For $t \leqslant x$ kernel $K(x, t)$ satisfies the integral equation

$$f(x,t) + \int_0^x K(x,s) f(s,t)\,ds + K(x,t) = 0$$

$$f(x,t) = \frac{\partial^2}{\partial x \partial t}\int_{-\infty}^{\infty} \gamma^{-1}\sin\sqrt{\gamma}x \sin\sqrt{\gamma}t\,d\tau(\gamma)$$

$$\sigma(\gamma) = \sum_{\gamma_k \leqslant \gamma} \frac{1}{\beta_k}, \quad \tau(\gamma) = \begin{cases} \sigma(\gamma) - \frac{2}{\pi}\gamma, & \gamma \geqslant 0 \\ \sigma(\gamma), & \gamma < 0 \end{cases}$$

where $\sigma(\gamma)$ is the spectral function of operator (1.9). Function $l(x)$ and the constant appearing in the boundary condition are determined in terms of kernel $K(x, t)$ by

$$l(x) = \frac{1}{2}\frac{dK(x,x)}{dx}, \quad \frac{\theta'(0)}{2\theta(0)} = K(0,0)$$

Function $\theta(x)$ can be expressed directly in terms of kernel $K(x, t)$, without having to integrate Eq. (1.10). Setting in (1.13) $\gamma = 0$, we obtain

$$\psi_0(x) = \psi(x,0) = 1 + \int_0^x K(x,t)\,dt \tag{1.14}$$

For $\gamma = 0$ function $\psi_0(x)$ satisfies Eq. (1.9) and initial conditions (1.12). But function $\sqrt{\theta(x)/\theta(0)}$ also satisfies that equation and the same boundary conditions. Hence

$$\theta(x) = \theta(0)\,\psi_0^2(x) \tag{1.15}$$

Taking into consideration (1.8) and (1.15), we obtain for $k(r)$ the parameteric expression

$$r(x) = \left(2\frac{B}{\pi}\theta(0)\int_0^x \psi_0^2(x)\,dx + r_0^2\right)^{1/2} \qquad (1.16)$$

$$k(x) = \theta^2(0)\,\psi_0^4(x)\,r^{-2}(x)$$

Function $\psi_0(x)$ is determined by formula (1.14) and $\theta(0)$ by equality (1.11).

Thus, if function $k(r) > 0$, $k(r) \in C^1[r_0, R]$ for which the specified conditions of borehole operation obtains, then in the class of positive continuously differentiable in $[r_0, R]$ functions $k(r)$ that function is unique and is determined by equality (1.16). In this case the overdetermined system of boundary conditions (1.2) and (1.3) is understood to represent the operation conditions.

We shall now derive certain integral relationships for function $\rho(r)$ which may be useful for an approximate determination of $\rho(r) - \rho(r_0)$. In the physical sense that function represents the filtration resistance of ring $r_0 \leqslant r_1 \leqslant r$ under stabilized pumping conditions.

It can be readily shown that functions $P_k(r)\sqrt{r}$, where $P_k(r)$ are eigenfunctions of operator (1.4), are eigenfunctions of the integral operator

$$P_k(r)\sqrt{r} = \lambda_k\int_{r_0}^{R} G_1(r, r_1)\,P_k(r_1)\sqrt{r_1}\,dr_1 \qquad (1.17)$$

$$\frac{1}{\sqrt{rr_1}}G_1(r, r_1) = -\begin{cases}\rho(r_1), & r_0 \leqslant r \leqslant r_1\\ \rho(r), & r_1 \leqslant r \leqslant R\end{cases}$$

We introduce the notation

$$G_p(r, r_1) = \int_{r_0}^{R} G_{p-1}(r, s)\,G_1(s, r_1)\,ds, \qquad p = 2, 3 \ldots$$

For the resolvent of kernel $G_1(r, r_1)$ the expansion

$$R(r, r_1, \lambda) = \sum_{k=1}^{\infty}\lambda^{k-1}G_k(r, r_1) \qquad (1.18)$$

is valid in the neighborhood $\lambda = 0$. On the other hand, $R(r, r_1, \lambda)$ may be represented as a series in eigenfunctions of operator (1.17)

$$R(r, r_1, \lambda) = \sum_{k=1}^{\infty}\frac{P_k(r)\,P_k(r_1)\sqrt{rr_1}}{\lambda_k - \lambda} \qquad (1.19)$$

Allowing for (1.7) and (1.18), for $r = r_1 = r_0$ we obtain

$$R(r_0, r_0, \lambda) = \sum_{k=1}^{\infty}\frac{r_0 P_k^2(r_0)}{\lambda_k - \lambda} = \sum_{k=1}^{\infty}\lambda^{k-1}G_k(r_0, r_0) = -\frac{\Phi^*(\lambda)\,r_0}{Q^*(\lambda)} \qquad (1.20)$$

Function $\Phi^*(\lambda)\,r_0\,/\,Q^*(\lambda)$ is known, hence coefficients

$$G_k(r_0, r_0) = a_k, \qquad k = 1, 2, 3\ldots \qquad (1.21)$$

are also known. The quantities $G_k(r_0, r_0)$ are expressed in terms of integrals of function $\rho(r)$.

Note that for the determination of

$$G_k(r_0, r_0) = \int_{r_0}^{R} G_{k-1}(r_0, s)\, G_1(s, r_0)\, ds \tag{1.22}$$

it is necessary to know the iterated kernels only as functions of one argument, for instance, of the second one (kernels $G_k\ (r,\ r_1)$ are symmetric).

Let us expand each function $G_k\ (r_0,\ r)$ over the complete ortho-normal system $h_i\ (r)$

$$-\sqrt{r_0 r}\,\rho(r) = G_1(r_0, r) = \sum_{i=1}^{\infty} c_i h_i(r) \tag{1.23}$$

$$G_k(r_0, r) = \sum_{i=1}^{\infty} c_{i,k} h_i(r)$$

The definition of iterated kernels implies that coefficients $c_{i,k}$ can be expressed in terms of c_i. Limiting expansion (1.23) to the first N terms with N unknown c_i, $i = 1, 2, \ldots N$, we derive a system of N algebraic equations in c_i. Using (1.22) and (1.23), we obtain

$$\sum_{i=1}^{N} c_i h_i(r_0) = a_1, \quad \sum_{i=1}^{N} c_i c_{i,k} = a_{k+1},\ k = 1, 2, \ldots, N-1 \tag{1.24}$$

Coefficients $c_{i,k}$ satisfy the recurrent relationships

$$c_{i,\,k+1} = \sum_{m,\,n=1}^{\infty} \alpha_{imn} c_{m,k} c_n$$

$$\alpha_{imn} = \frac{1}{\sqrt{r_0}} \left\{ \int_{r_0}^{R} h_i h_n \left(\int_{r_0}^{r} h_m \sqrt{r_1}\, dr_1 \right) dr + \int_{r_0}^{R} h_i \sqrt{r} \left(\int_{r}^{R} h_m h_n\, dr_1 \right) dr \right\}$$

Although the solution of system (1.24) is not unique, it is possible to overdetermine it by making $k \geqslant N$ and solving the overdetermined system by some approximate method.

Note that, when test pumping is carried out in a borehole which had previously been worked under steady conditions, then instead of (1.7) we have

$$\sum_{k=1}^{\infty} \frac{P_k^2(r_0)}{\lambda_k - \lambda} = -\frac{\Phi^*(\lambda) + p_0/\lambda}{Q^*(\lambda) + q_0/2\pi\lambda}$$

where q_0 and p_0 are, respectively, the pumping rate and pressure in the borehole under steady working conditions.

2. Correctness of the determination of the permeability coefficient by the flow rate and pressure in the borehole. Let us examine the stability of determination of function $k\ (r)$ for small perturbations $\varphi\ (t)$ and $q\ (t)$.

It is well known that problems of this kind are not precise in the classical sense, since small variations of input data can considerably affect the solution of the related inverse problem. However with some *a priori* restrictions imposed on function $k\ (r)$ it is possible to prove the nominal stability of the considered problem.

It follows from (1.16) that the problem of determination of $k\ (r)$ by $\varphi\ (t)$ and $q\ (t)$ is proper, if the problem of finding function $\psi_0\ (x)$ is proper.

It was shown in [8] that, if the two spectral functions $\sigma_1(\gamma)$ and $\sigma_2(\gamma)$ in problem (1.9) coincide for $-\infty < \gamma < N$ and

$$|h| < d, \qquad |l(x)| < D \tag{2.1}$$

then for $\psi_0(x)$ we have the inequality

$$\max_{0\leqslant x\leqslant\pi} |\psi_{01}(x) - \psi_{02}(x)|^2 \leqslant \frac{S\exp\{2N^{-1/2}(d + DN^{-1/2})\}}{N^{1/2}[1 - N^{-1/2}(d + \pi D)]^2} \tag{2.2}$$

for $\sqrt{N} > d + \pi D$, where S is some constant dependent on d and D.

It follows from (2.2) that

$$\max_{0\leqslant x\leqslant\pi} |\psi_{01}(x) - \psi_{02}(x)| \to 0 \quad \text{for} \quad N \to \infty \tag{2.3}$$

An inequality similar to (2.2) can be also derived in the case in which the spectral function along the interval $(-\infty, N)$ is known with an error δ. Equation (2.3) is valid for $\delta \to 0$.

We select some function $q(t)$ for which the Laplace transformation exists. We denote by H the class of functions $k(r)$ such that

$$k(r) \in C^2[r_0, R], \quad 0 < a \leqslant k(r) \leqslant b < \infty$$
$$|k'(r)| \leqslant M_1, \qquad |k''(r)| \leqslant M_2$$

Let A_q be an operator which brings about the congruence between any $k(r) \in H$ and function $\varphi(t) = p(r_0, t)$, where $p(r, t)$ is the solution of Eq. (1.1) with boundary conditions (1.2).

We introduce the following norms for functions dependent on t and r, respectively:

$$\|\varphi(t)\| = \max_{t\in(0,\infty)} |\varphi(t)|, \qquad \|k(r)\| = \max_{r\in[r_0, R]} |k(r)|$$

Unless otherwise stated, an arrow will subsequently denote convergence for $n \to \infty$.

We denote by U_q the set of functions

$$\varphi(t) = A_q k(r), \qquad k(r) \in H$$

Then, as previously shown, the inverse operator $A_q^{-1}\varphi(t) = k(r)$ is determinate on U_q.

Let us show the correctness by Tikhonov's criterion of the inverse problem in the class of functions $k(r) \in H$, i.e. that when

$$\|\varphi(t) - \varphi_n(t)\| \to 0, \qquad \varphi, \varphi_n \in U_q$$

then

$$\|A_q^{-1}\varphi - A_q^{-1}\varphi_n\| = \|k(r) - k_n(r)\| \to 0$$

To do this it is sufficient to show, as implied by (2.2), that for $\|\varphi - \varphi_n\| \to 0$ the following conditions are satisfied:

1) the inequalities

$$|h| < d, \qquad |l(x)| < D$$

are valid for problem (1.9)

2) for any k

$$\lambda_{kn} \to \lambda_k, \ \alpha_{kn} \to \alpha_k, \ B_n \to B, \ \theta_n(0) \to \theta(0)$$

where λ_{kn}, α_{kn}, B_n and $\theta_n(0)$ are the related parameters of problem (1.4) with the permeability coefficient $k_n(r)$.

Condition (1) can be satisfied by transforming Eq. (1.4) into (1.9). In that case constants d and D are common for all $k(r) \in H$ and are expressed in terms of a, b, M_1 and M_2.

Following the idea of [8] and taking into consideration the restrictions on set H, it is possible to prove that for $k(r) \in H$ the eigenvalues λ_k and the normalization factors α_k of operator (1.4) have the following properties:

1) λ_k and α_k tend on H uniformly to asymptotics, i.e.

$$\sqrt{\lambda_k} = \frac{\pi}{B}\left(k + \frac{1}{2}\right) + \frac{\pi}{B}\frac{\gamma_k}{k}$$

$$\alpha_k = \frac{B\theta(0)}{2} + \frac{B\theta(0)}{\pi}\frac{\eta_k}{k}$$

$$\sup_{k,\, k(r)\in H} \gamma_k < \gamma_0 < \infty, \qquad \sup_{k,\, k(r)\in H} \eta_k < \eta_0 < \infty$$

2) There exist constants λ', λ'' and α', α'' ($\lambda', \alpha' > 0$) such that for any k, $k(r) \in H$

$$\lambda' \leqslant \lambda_1 \leqslant \lambda'', \qquad \alpha' \leqslant \alpha_k \leqslant \alpha''$$

3)

$$\inf_{k,\, k(r)\in H} |\lambda_k - \lambda_{k-1}| \geqslant \Delta\lambda > 0$$

Let us prove Condition (2). We set in correspondence to every function $k(r) \in H$ the function

$$f(t) = \sum_{k=1}^{\infty} \frac{1}{\beta\alpha_k} e^{-\lambda_k t/\beta} \tag{2.4}$$

where $\{\alpha_k, \lambda_k\}$ are spectral parameters of operator (1.4). We denote the set of functions of the form (2.4) by $F = \{f\}$. If $\varphi(t) \in U_q$, it follows from (1.6) that

$$\varphi(t) = -\frac{1}{2\pi}\int_0^t q(t-\tau) f(\tau)\, d\tau = -\frac{1}{2\pi} q(t) * f(t), \qquad f(t) \in F$$

Let the sequence

$$\varphi_n(t) = -\frac{q(t)}{2\pi} * f_n(t) = -\frac{q(t)}{2\pi} * \sum_{k=1}^{\infty} \frac{1}{\beta\alpha_{kn}} c^{-\lambda_{kn} t/\beta} \tag{2.5}$$

where λ_{kn} and α_{kn} are spectral parameters of operator (1.4), for $k_n(r) \in H$), converge to $\varphi(t)$.

Let us prove that $\lambda_{1n} \to \lambda_1$ and $\alpha_{1n} \to \alpha_1$, and assume that the opposite is true. Then there must exist a sequence $\{f_n'\} \subset \{f_n\}$ such that

$$\lambda_1 < \lambda_{1n}' - \rho \tag{2.6}$$

or

$$\lambda_{1n}' + \rho < \lambda_1 \qquad (\rho > 0) \tag{2.7}$$

We consider inequality (2.6). In that case the remainder $\Delta f_n' = f - f_n'$ for reasonably great t is greater than some positive function independent of n . In fact,

$$\Delta f_n' \geqslant e^{-\lambda_1 t/\beta}\left(\frac{1}{\beta\alpha_1} - \sum_{k=1}^{\infty} \frac{1}{\beta\alpha_{kn}'} e^{-(\lambda_{kn}' - \lambda_1)t/\beta}\right)$$

By virtue of inequality (2.6) and properties (1) and (2) there exists a $\tau_0 > 0$ such that for $t \geqslant \tau_0$ the inequality

$$\Delta f_n' \geqslant \frac{1}{2\beta\alpha_1} e^{-\lambda_1 t/\beta} \qquad (2.8)$$

is satisfied by all f_n'. Let us show that the last inequality contradicts the condition of convergence of functions $\varphi_n'(t)$ to $\varphi(t)$. For this we investigate the sequence of widening segments

$$[t_m, T_m], \qquad t_m \to 0, \qquad T_m \to \infty, \qquad m = 1, 2\ldots,$$

It follows from (1) and (2) that along any segment $[t, T]$, $0 < t < T < \infty$ the set of functions F is uniformly bounded and equicontinuous, hence the set F is a compact in $[t, T]$.

There exists then function Δf_0 and the chain of sequences

$$\{\Delta f_n'\} \supset \{\Delta f_{n1}'\} \supset \ldots \supset \{\Delta f_{nm}'\} \supset \ldots$$

such that along segment $[t_m, T_m]$ function $\Delta f_{nm}'$ uniformly converges to function Δf_0.

Let t_0, $t \in [t_m, T_m]$. Then

$$\varphi(t) - \varphi_{nm}'(t) = \int_0^{t_0} \Delta f_{nm}'(\tau)\, q(t-\tau)\, d\tau + \qquad (2.9)$$

$$\int_{t_0}^{t} (\Delta f_{nm}'(\tau) - \Delta f_0(\tau))\, q(t-\tau)\, d\tau + \int_{t_0}^{t} \Delta f_0(\tau)\, q(t-\tau)\, d\tau$$

Since function $\Delta f_{nm}'(t)$ is integrable in the neighborhood of zero when $n \to \infty$, $t_m \to 0$ and $t_0 \to 0$, it follows from (2.9) that

$$\int_0^t \Delta f_0(\tau)\, q(t-\tau)\, d\tau \equiv 0, \qquad \Delta f_0(t) \equiv 0$$

But the last equality contradicts (2.8). In exactly the same way it is possible to show the contradiction with (2.8) by analyzing the case of (2.7) or assuming that α_{1n} does not converge to α_1. Then, using property (3), by similar reasoning, we obtain

$$\lambda_{kn} \to \lambda_k, \qquad \alpha_{kn} \to \alpha_k \text{ for } \forall\, k$$

But then from (1) follows that $B_n \to B$ and $\theta_n(0) \to \theta(0)$.

Thus Conditions (1) and (2) are satisfied and, consequently, the input problem is proper in the class of $k(r) \in H$.

More generally, it is possible to consider the space W of pairs of functions $\langle q, \varphi \rangle$, where q is an arbitrary function for which the Laplace transformation $\varphi \in U_q$ exists. We define the convergence in W as follows:

$$\langle q_n, \varphi_n \rangle \to \langle q, \varphi \rangle, \qquad \text{if } q_n \to q \text{ and } \varphi_n \to \varphi$$

Then operator A^{-1} which maps W in H (not one-to-one) is continuous in W.

The proof is similar to the above.

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ON THE MOTION OF GROUND WATERS IN A STRATUM BETWEEN TWO SLIGHTLY PERMEABLE STRATA UNDER CONDITIONS OF PRESSURE-NO-PRESSURE HEAD

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The problem of motion of ground waters in a stratum under conditions of pressure-no-pressure and seepage through the upper and lower strata is considered. This problem reduces to solving a system of functional equations whose solution can be obtained by the method of successive approximations. The behavior of the unknown boundary $x = x_1(t)$ which separates regions of pressure and no-pressure motion is investigated. Proof is given of the uniqueness of the solution around $t = t_0$.

Let us consider the problem of finding a differentiable function $x_1(t)$ such that $x_1(t_0) = 0$ and $x_1(t) > 0$ for $t \in [t_0, T]$, and the solution $u_i(x, t)$ (with continuous derivative u_{ix}) of equation

$$\frac{\partial u_i}{\partial t} = a_i \frac{\partial^2 u_i}{\partial x^2} - b_i (u_i - u_{0i}) \tag{1}$$

in region

$$\Omega_1 = \{(t, x): 0 < x < x_1(t), t_0 < t \leqslant T\} \quad \text{for } i = 1$$

$$\Omega_2 = \{(t, x): x_1(t) < x < \infty, t_0 < t \leqslant T\} \quad \text{for } i = 2$$

which satisfies conditions

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=+0} = q(t), \quad u_2|_{t=t_0+0} = \varphi(x), \quad u_2|_{x\to+\infty} = \text{const} \tag{2}$$

$$u_1|_{x=x_1(t)-0} = u_0^-, \quad u_2|_{x=x_1(t)+0} = u_0^+, \quad \left.\frac{\partial u_1}{\partial x}\right|_{x=x_1(t)-0} = \left.\frac{\partial u_2}{\partial x}\right|_{x=x_1(t)+0} \tag{3}$$